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Well-balanced finite volume evolution Galerkin methods for the shallow water equations with source terms

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SUMMARY

The goal of this paper is to present a new well-balanced genuinely multi-dimensional high-resolution finite volume evolution Galerkin method for systems of balance laws. The derivation of the method will be illustrated for the shallow water equation with geometrical source term modelling the bottom topography. The results can be generalized to more complex systems of balance laws. Copyright © 2005 John Wiley & Sons, Ltd.

KEY WORDS: well-balanced methods; finite-volume evolution Galerkin schemes; shallow water equations; truly multi-dimensional finite-volume methods

1. INTRODUCTION

We consider the balance law in two space dimensions

$$\mathbf{u}_t + \mathbf{f}_1(\mathbf{u})_x + \mathbf{f}_2(\mathbf{u})_y = \mathbf{b}(\mathbf{u}, x, y) \tag{1}$$

where $\mathbf{u}$ stands for the vector of the conservative variables, $\mathbf{f}_1, \mathbf{f}_2$ are flux functions and $\mathbf{b}(\mathbf{u}, x, y)$ is a source term. For the case of homogenous conservation laws, i.e. $\mathbf{b}(\mathbf{u}, x, y) = 0$, several high-resolution genuinely multi-dimensional schemes have been developed in the literature, see, e.g. References [1–3]. In this paper, we are concerned with the finite volume evolution Galerkin (FVEG) method of Lukáčová, Morton and Warnecke, cf. References [4–7]. The FVEG methods couple a finite volume formulation with approximate evolution operators which are based on the theory of bicharacteristics for the first-order systems [5]. As a result, exact integral equations for linear or linearized hyperbolic conservation laws can be derived.

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They take all of the infinitely many directions of wave propagation into account. For two-dimensional conservation laws, this is realized by the integration along the sonic circle, i.e. for a parameter $\theta \in [0, 2\pi]$. Further integrals appearing in the exact integral equations are the integrals along time, e.g. from t_n to t_{n+1}. Since the exact integral equations are implicit in time, appropriate numerical quadratures have to be applied in time in order to approximate integrals along the mantle of the so-called bicharacteristic cones. This yields the approximate evolution operators that are explicit in time. In the finite volume framework, the approximate evolution operators are used to evolve the solution along the cell interfaces in order to compute fluxes on edges. This step can be considered as a predictor step. In the corrector step the finite volume update is done. In summary, the FVEG scheme is a genuinely multi-dimensional method that is explicit in time. The error analysis of the FVEG schemes was studied theoretically for the linearized systems of hyperbolic conservation laws in Reference [5]. New approximate evolution operators developed in Reference [6] improved stability of the whole finite volume EG scheme, see also Reference [7]. It has been shown in Reference [6] that the new FVEG scheme has not only enlarged the area of stability but it is also considerably more accurate than other commonly used FV schemes. Relatively high global accuracy of the FVEG schemes has been confirmed in general by extensive numerical treatment in series of papers References [4–6] for linear as well as non-linear conservation laws.

For balance laws with source terms, the simplest approach is to use the operator splitting method which alternates between the homogenous conservation laws $\mathbf{u}_t + \mathbf{f}_1(\mathbf{u})_x + \mathbf{f}_2(\mathbf{u})_y = 0$ and the ordinary differential equation $\mathbf{u}_t = \mathbf{b}(\mathbf{u}, x, y)$ in each time step. For many situations, this would be effective and successful. However, the original problem (1) has an interesting structure, which is due to the competition between the differential terms and the right-hand side source term during the time evolution. If we split *a priori* these terms, which are dominant for the evolution process, numerical schemes can yield spurious solutions. In particular, the equilibrium or stationary states, i.e. $\mathbf{u}$ such that

$$\mathbf{f}_1(\mathbf{u})_x + \mathbf{f}_2(\mathbf{u})_y = \mathbf{b}(\mathbf{u}, x, y)$$

cause difficulties. These equilibrium solutions usually play an important role because they are obtained as a limit when time tends to infinity.

In this paper, we present an approach which allows to incorporate treatment of the source in the framework of the FVEG schemes without using the operator splitting approach. Thus, the stationary states, or quasi-stationary states, will be approximated correctly. The scheme is called the *well-balanced finite volume evolution Galerkin scheme*; see also, e.g. Reference [8] (cf. the C property) and Reference [9] for other related approaches in the literature.

2. SHALLOW WATER EQUATIONS AND THE WELL-BALANCED APPROXIMATE EVOLUTION OPERATORS

There are many practical applications where the balance laws and the correct approximation of their quasi-steady states are needed. In what follows, we illustrate the methodology on the example of the shallow water equations with the bottom topography term.

This system reads

$$\mathbf{u}_t + \mathbf{f}_1(\mathbf{u})_x + \mathbf{f}_2(\mathbf{u})_y = \mathbf{b}(\mathbf{u}) \tag{2}$$

where

$$\mathbf{u} = \begin{pmatrix} h \\ hu \\ hv \end{pmatrix}, \quad \mathbf{f}_1(\mathbf{u}) = \begin{pmatrix} hu \\ hu^2 + \frac{1}{2}gh^2 \\ huv \end{pmatrix}, \quad \mathbf{f}_2(\mathbf{u}) = \begin{pmatrix} hv \\ huv \\ hv^2 + \frac{1}{2}gh^2 \end{pmatrix}, \quad \mathbf{b}(\mathbf{u}) = \begin{pmatrix} 0 \\ -ghb_x \\ -ghb_y \end{pmatrix}$$

Here h denotes the water depth, u and v are vertically averaged velocity components in x- and y-direction, g stands for the gravitational constant and $b = b(x, y)$ denotes the bottom topography. It should be pointed out that for practical problems, for example the river or oceanographic flows, some additional terms modelling the bottom friction or the Coriolis forces need to be considered as well. Applying the theory of bicharacteristics to (2) leads to the integral equations in an analogous way as in Reference [5]

$$\begin{aligned} h(P) = {} & \frac{1}{2\pi} \int_0^{2\pi} h(Q) - \frac{\tilde{c}}{g} u(Q) \cos\theta - \frac{\tilde{c}}{g} v(Q) \sin\theta \, \mathrm{d}\theta \\ & - \frac{1}{2\pi} \int_{t_n}^{t_{n+1}} \frac{1}{t_{n+1} - \tilde{t}} \int_0^{2\pi} \frac{\tilde{c}}{g} (u(\tilde{Q}) \cos\theta + v(\tilde{Q}) \sin\theta) \, \mathrm{d}\theta \, \mathrm{d}\tilde{t} \\ & + \frac{1}{2\pi} \tilde{c} \int_{t_n}^{t_{n+1}} \int_0^{2\pi} (b_x(\tilde{Q}) \cos\theta + b_y(\tilde{Q}) \sin\theta) \, \mathrm{d}\theta \, \mathrm{d}\tilde{t} \end{aligned} \tag{3}$$

$$\begin{aligned} u(P) = {} & \frac{1}{2} u(Q_0) + \frac{1}{2\pi} \int_0^{2\pi} - \frac{g}{\tilde{c}} (h(Q) + b(Q)) \cos\theta + u(Q) \cos^2\theta + v(Q) \sin\theta \cos\theta \, \mathrm{d}\theta \\ & - \frac{1}{2} g \int_{t_n}^{t_{n+1}} (h_x(\tilde{Q}_0) - b_x(\tilde{Q}_0)) \, \mathrm{d}\tilde{t} \\ & - \frac{1}{2\pi} g \int_{t_n}^{t_{n+1}} \int_0^{2\pi} (b_x(\tilde{Q}) \cos^2\theta + b_y(\tilde{Q}) \cos\theta \sin\theta) \, \mathrm{d}\theta \, \mathrm{d}\tilde{t} \\ & + \frac{1}{2\pi} \int_{t_n}^{t_{n+1}} \frac{1}{t_{n+1} - \tilde{t}} \int_0^{2\pi} (u(\tilde{Q}) \cos 2\theta + v(\tilde{Q}) \sin 2\theta) \, \mathrm{d}\theta \, \mathrm{d}\tilde{t} \end{aligned} \tag{4}$$

with an analogous equation for the second velocity component v. Here $P = (x, y, t_{n+1})$ is the pick of the bicharacteristic cone, $Q_0 = (x - \tilde{u}\Delta t, y - \tilde{v}\Delta t, t_n)$ denotes the centre of the sonic circle, $\tilde{Q}_0 = (x - \tilde{u}(t_n + \Delta t - \tilde{t}), y - \tilde{v}(t_n + \Delta t - \tilde{t}), \tilde{t})$, $\tilde{Q} = (x - \tilde{u}(t_n + \Delta t - \tilde{t}) + c(t_n + \Delta t - \tilde{t}) \cos\theta, y - \tilde{v}(t_n + \Delta t - \tilde{t}) + c(t_n + \Delta t - \tilde{t}) \sin\theta, \tilde{t})$ stays for arbitrary point on the mantle and $Q = Q(\tilde{t})|_{\tilde{t} = t_n}$ denotes a point at the perimeter of the sonic circle at time t_n. The local velocities are denoted by $\tilde{u}, \tilde{v}, \tilde{c} = \sqrt{g\tilde{h}}$.

Lemma 2.1
The well-balanced approximation of the integral equations (3), (4) reads

$$
\begin{aligned}
h(P) = {} & -b(P) + \frac{1}{2\pi}\int_0^{2\pi} (h(Q)+b(Q)) - \frac{\tilde{c}}{g}u(Q)\cos\theta - \frac{\tilde{c}}{g}v(Q)\sin\theta \,\mathrm{d}\theta \\
& -\frac{1}{2\pi}\int_{t_n}^{t_{n+1}} \frac{1}{t_{n+1}-\tilde{t}} \int_0^{2\pi} \frac{\tilde{c}}{g}(u(\tilde{Q})\cos\theta + v(\tilde{Q})\sin\theta)\,\mathrm{d}\theta\,\mathrm{d}\tilde{t} + O(\Delta t^2)
\end{aligned} \tag{5}
$$

$$
\begin{aligned}
u(P) = {} & \frac{1}{2}u(Q_0) + \frac{1}{2\pi}\int_0^{2\pi} -\frac{g}{\tilde{c}}(h(Q)+b(Q))\cos\theta + u(Q)\cos^2\theta + v(Q)\sin\theta\cos\theta\,\mathrm{d}\theta \\
& -\frac{1}{2\pi}\frac{g}{\tilde{c}}\int_{t_n}^{t_{n+1}} \frac{1}{t_{n+1}-\tilde{t}}\int_0^{2\pi} (h(\tilde{Q})+b(\tilde{Q}))\cos\theta\,\mathrm{d}\theta\,\mathrm{d}\tilde{t} \\
& +\frac{1}{2\pi}\int_{t_n}^{t_{n+1}} \frac{1}{t_{n+1}-\tilde{t}}\int_0^{2\pi} (u(\tilde{Q})\cos 2\theta + v(\tilde{Q})\sin 2\theta)\,\mathrm{d}\theta\,\mathrm{d}\tilde{t} + O(\Delta t^2)
\end{aligned} \tag{6}
$$

with an analogous equations for the second velocity v.

Lemma 2.1 can be proved by applying the trapezoidal rule for time integrals and the Taylor expansion as well as the Gauss theorem on the sonic circle. Approximations (5) and (6) are well balanced in the sense that the steady equilibrium states, i.e. $u, v = 0, h + b = \text{const.}$ are preserved; see Reference [10] for details of the proof. An important property of the evolution operator (5) and (6) is that the bottom elevation and the depth of the water are represented by the same terms. The next step is to approximate appropriately the mantle integrals, i.e. time integrals from t_n to t_{n+1}. This is done by means of the numerical quadratures which were proposed in Reference [6] in such a way that any planar one-dimensional wave is calculated exactly. These quadratures are now used systematically for approximation of all mantle integrals, i.e. integrals $\int_{t_n}^{t_{n+1}}\int_0^{2\pi}$. Following Reference [6] we get the *well-balanced approximate evolution operator* $E_\Delta^{\mathrm{const}}$ for piecewise constant functions

$$
\begin{aligned}
h(P) = {} & -b(P) + \frac{1}{2\pi}\int_0^{2\pi} (h(Q)+b(Q)) - \frac{\tilde{c}}{g}u(Q)\,\mathrm{sgn}(\cos\theta) \\
& -\frac{\tilde{c}}{g}v(Q)\,\mathrm{sgn}(\sin\theta)\,\mathrm{d}\theta + O(\Delta t^2) \\
u(P) = {} & \frac{1}{2\pi}\int_0^{2\pi} -\frac{g}{\tilde{c}}(h(Q)+b(Q))\,\mathrm{sgn}(\cos\theta) + u(Q)\left(\cos^2\theta + \frac{1}{2}\right) \\
& + v(Q)\sin\theta\cos\theta\,\mathrm{d}\theta + O(\Delta t^2)
\end{aligned} \tag{7}
$$

The approximate evolution $E_\Delta^{\mathrm{bilin}}$ for bilinear functions can be derived from (5) and (6) in an analogous way as in Reference [6].

3. FINITE VOLUME EVOLUTION GALERKIN SCHEME

The above approximate evolution operators will now be used in the finite volume method in order to compute fluxes on cell interfaces. Let us consider for simplicity a regular rectangular mesh. The finite volume evolution Galerkin scheme for balance laws reads

$$\mathbf{U}^{n+1} = \mathbf{U}^n - \frac{\Delta t}{\Delta x}\sum_{k=1}^{2}\delta_{x_k}\mathbf{f}_k(\mathbf{U}^{n+1/2}) + \mathbf{B}^{n+1/2} \quad \mathbf{f}_k(\mathbf{U}^{n+1/2}) = \frac{1}{h}\int_{\mathscr{E}}\mathbf{f}_k(E_{\Delta t/2}\mathbf{U}^n)\,\mathrm{d}S \tag{8}$$

where $\mathbf{B}^{n+1/2}$ stays for the approximation of the source term, $\delta_{x_k}\mathbf{f}_k(\mathbf{U}^{n+1/2})$ represents an approximation to the edge flux difference at the intermediate time level $t_n + \Delta t/2$. The cell interface fluxes $\mathbf{f}_k(\mathbf{U}^{n+1/2})$ are evolved using an approximate evolution operator denoted by $E_{\Delta t/2}$ to $t_n + \Delta t/2$ and averaged along the cell interface edge denoted by $\mathscr{E}$.

For the first-order scheme, the approximate evolution operator $E_{\Delta t/2}^{\text{const}}$ for the piecewise constant data is used. For the second-order method, the continuous bilinear recovery R_h is applied first. Then the predicted solution at cell interfaces is obtained in the following way:

$$\mathbf{f}_k(\mathbf{U}^{n+1/2}) = \frac{1}{h}\int_{\mathscr{E}}\mathbf{f}_k(E_{\Delta t/2}^{\text{bilin}}R_h\mathbf{U}^n + E_{\Delta t/2}^{\text{const}}(1-\mu_x^2\mu_y^2)\mathbf{U}^n)\,\mathrm{d}S \tag{9}$$

where $\mu_x^2 U_{ij} = \frac{1}{4}(U_{i+1,j} + 2U_{ij} + U_{i-1,j})$; an analogous notation is used for the y-direction.

The source term $\mathbf{B}$ will be approximated in the so-called interface-based way in order to reflect a delicate balance between the gradient of flux functions and the right-hand side source term for quasi-steady stationary states. In fact, we have for nearly hydrostatic flows that $\sqrt{u^2+v^2} \ll \sqrt{gh}$. In the associated asymptotic limit, the leading order water height h satisfies the balance of momentum flux and momentum source terms. More precisely, we have from the momentum equation in x-direction $\partial_x(gh^2/2) = -ghb_x$. This is the condition that yields the well-balanced approximation of the source term. Integrating, e.g. in the second equation of (2), the right-hand side over the mesh cell Ω_{ij} we get

$$\begin{aligned}\frac{1}{\Delta x^2}\int_{\Omega_{ij}} B_2(\mathbf{U}^{n+1/2}) &= \frac{1}{\Delta x^2}\int_{x_{i-1/2}}^{x_{i+1/2}}\int_{y_{i-1/2}}^{y_{i+1/2}} -gh^{n+1/2}b_x \\ &\approx \frac{-g}{\Delta x}\int_{y_{i-1/2}}^{y_{i+1/2}} \frac{h_{i+1/2}^{n+1/2} + h_{i-1/2}^{n+1/2}}{2}\,\frac{b_{i+1/2} - b_{i-1/2}}{\Delta x}\end{aligned}$$

It is easy to see that for quasi-steady stationary cases, i.e. $h + b \approx \text{const.}$, the latter term is equivalent to the flux differences on cell interfaces which arise from the finite-volume update.

4. NUMERICAL EXPERIMENT

In the following experiment, we have tested the resolution of small perturbations of steady states. The bottom topography consists of one hump

$$b(x) = \begin{cases} 0.25(\cos(10\pi(x-0.5))+1) & \text{if } |x-0.5| < 0.1 \\ 0 & \text{otherwise} \end{cases}$$

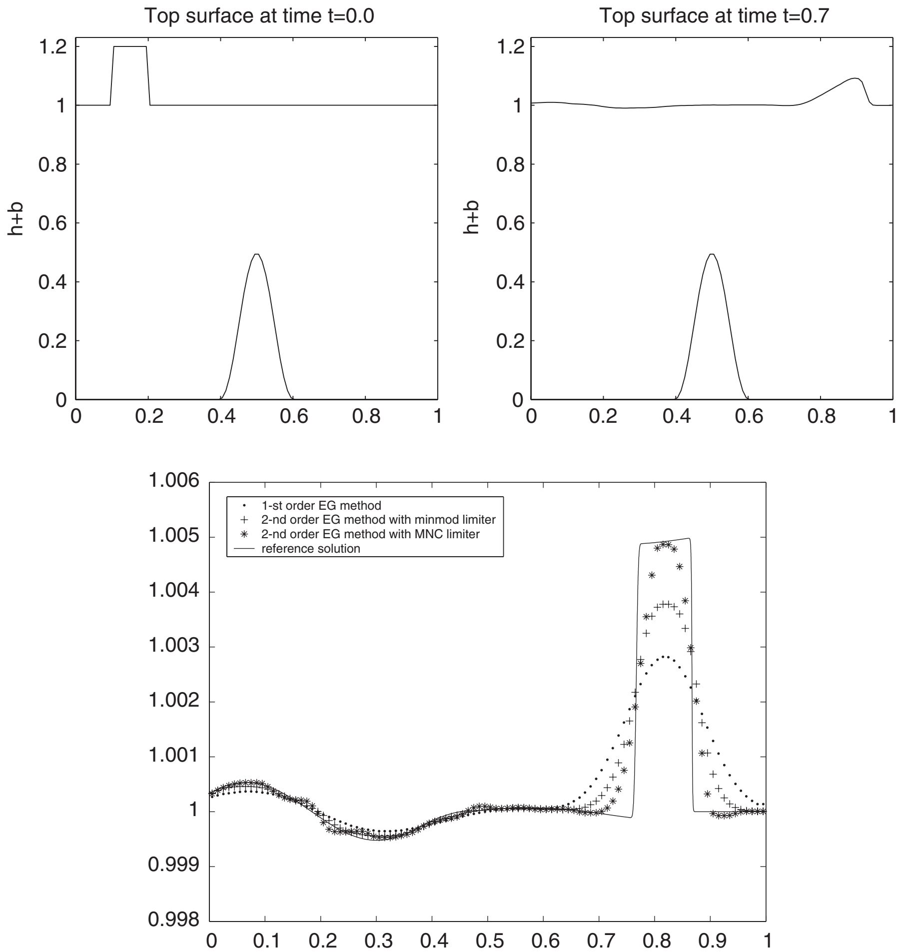

Figure 1. Propagation of small perturbances, $\varepsilon = 0.2$ (top) and a magnified view for $\varepsilon = 0.01$ (bottom).

and the initial data are $u(x, 0) = 0$,

$$h(x,0) = \begin{cases} 1 - b(x) + \varepsilon & \text{if } 0.1 < x < 0.2 \\ 1 - b(x) & \text{otherwise} \end{cases}$$

The parameter ε is chosen to be 0.2 or 0.01. The computational domain is $[0, 1]$ and the extrapolation boundary conditions have been used.

In Figure 1, we can see propagation of small perturbances of the water depth h until time $t = 0.7$. The solution is computed on a mesh with 100 cells. In the top picture, the parameter of perturbation $\varepsilon = 0.2$ is relatively large in comparison to the discretization error. In the bottom picture $\varepsilon = 0.01$. The solution is computed with the first- and the second-order FVEG methods using the minmod and the monotonized minmod limiters. The reference solutions were obtained by the second-order FVEG method with the minmod limiter on a mesh with

10 000 cells. We can notice correct resolution of small perturbances of the steady state even if the perturbances are of the order of the truncation error. Similar results have been obtained also for two-dimensional problems.

The present research is in progress. We are currently studying the approximation of dry states, i.e. $h \approx 0$, as well as other quasi-steady states where the momentums hu, hv are non-zero constants. The behaviour of such flow depends on the bottom topography and on the free-stream Froude number $Fr = \sqrt{u^2 + v^2}/\sqrt{gh}$. For intermediate Fr, the flow can be transcritical and the solution can contain a stationary transcritical shock.

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